Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 4, pp. 512–518, 2002

WAVES ON THE FREE SURFACE OF A TWO-PHASE MEDIUM

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UDC 532.59: 532.547

A boundary-value problem is posed to determine the wave motion caused by propagation of a gravity wave on the free surface of a layer of a two-phase medium. The problem is solved analytically in the linear approximation. The shape of the free surface, the phase velocity, and the frequency and damping factor of the wave are determined. An example of the solution of the problem is given.

Investigation of propagation of surface waves over a layer of a two-phase (disperse) liquid is of both theoretical and practical interest. The results of these investigations can be used to study the influence of admixtures on wave parameters. This influence on propagation of near-shore waves is described in [1]. At the same time, the study of the wave motion of a disperse liquid mixture allows one to extend the area of applicability of the theory of surface waves and also the dynamic theory of two-phase media. Lobov et al. [2] examined standing monochromatic waves on the interface of liquid layers and a mixture of this liquid with solid particles in the linear approximation and numerically determined the conditions of stability of this surface.

The objective of the present work is to study the influence of admixtures on wave propagation over the free surface of a two-phase medium.

1. Physical Model. We consider a layer of a disperse liquid mixture of constant thickness, which is located on a solid horizontal base. From above, the layer borders upon a medium of negligibly low density, which is characterized by a constant pressure $P_{\rm atm}$ (in particular, atmospheric). It is assumed that the carrier phase is an ideal incompressible liquid whose viscosity can be manifested only on the interface; the disperse phase consists of undeformable particles of identical size. There is no heat and mass transfer through the free surface and between the phases. The motion of such a two-phase medium is described by two-velocity equations of conservation of mass and momentum [3]:

$$\frac{\partial\rho_i}{\partial t}+\operatorname{div}(\rho_i\boldsymbol{v}_i)=0,\qquad \rho_i\,\frac{d\boldsymbol{v}_i}{dt}=-\alpha_i\nabla P_i+(-1)^iR\alpha_1\alpha_2(\boldsymbol{v}_1-\boldsymbol{v}_2)+\rho_i\boldsymbol{g},$$
$$\alpha_1+\alpha_2=1,\qquad \rho_i=\alpha_i\rho_i^0,\qquad \rho_i^0=\text{const},\quad i=1,2. \tag{1.1}$$

Here the subscripts $i=1$ and 2 refer to quantities that characterize the carrier and disperse phases, respectively, α_i, $\boldsymbol{v}_i$, P_i, ρ_i, and ρ_i^0 are the volume concentration, velocity, pressure, and reduced and true densities of the ith phase, respectively, and $\boldsymbol{g}$ is the acceleration of gravity. The coefficient $R(a,\eta)$ characterizes the interphase interaction. For instance, if we take into account only the Stokes force of viscous friction, we have $R=æ\eta/a^2$, where η is the dynamic viscosity of the liquid, a is the characteristic particle size, and $æ$ is the empirical coefficient of interphase friction (in the case of spherical particles, $æ=9/2$). If several forces of interphase interaction are taken into account, the coefficient R is equal to the sum of the corresponding coefficients.

We introduce a Cartesian coordinate system where the undisturbed surface coincides with the plane $z=0$. The bottom surface is $z=-l$ (l is the thickness of the layer of the mixture); the z axis and the vector $\boldsymbol{g}$ have the opposite directions. For system (1.1) to describe the wave motion of the mixture, it should be written for reduced pressures of the phases: pressure perturbations caused by wave propagation [4]. Assuming that the medium is at rest in the absence of the wave and using Eqs. (1.1) for $\boldsymbol{v}_i=0$ and the condition of equal pressures at the undisturbed surface $z=0$, we obtain the hydrostatic components of phase pressures $P_{i0}=P_{\rm atm}-\rho_i^0gz$. We assume

Tymen' State University, Tyumen' 625003. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 4, pp. 27–35, July–August, 2002. Original article submitted July 3, 2001; revision submitted February 13, 2002.

 0021-8944/02/4304-0512 $27.00

that phase-pressure perturbations caused by wave propagation over the free surface are identical in both phases. Then, the phase pressures are determined as

$$P_i = P_{i0} + p' = P_{\rm atm} - \rho_i^0 gz + p' \qquad (i = 1,\ 2), \tag{1.2}$$

where p' is the pressure perturbation due to the wave. Substituting (1.2) into system (1.1) and taking into account that ρ_i^0 is constant, we obtain a system of equations that describe the wave motion of the mixture:

$$\frac{\partial \alpha_i}{\partial t} + {\rm div}\,(\alpha_i \boldsymbol{v}_i) = 0,$$
$$\alpha_i \rho_i^0 \frac{d\boldsymbol{v}_i}{dt} = -\alpha_i \nabla p' + (-1)^i R\alpha_1\alpha_2(\boldsymbol{v}_1 - \boldsymbol{v}_2), \qquad \alpha_1 + \alpha_2 = 1. \tag{1.3}$$

The theory of surface waves for one-phase liquids imposes kinematic and dynamic conditions on the free surface $z = \xi(t, x, y)$ [4]. These condition follow from the absence of the mass flux through the surface and continuous momentum flux, respectively. We write these conditions for the case considered. The absence of the mass flux through the part of the free surface $z = \alpha_i(t, x, y, \xi)\xi(t, x, y)$ occupied by the ith phase is written as [3]

$$\alpha_i \rho_i^0 (v_{i,{\rm n}} - V_{\rm n}) = 0 \quad \text{or} \quad \alpha_i (v_{i,{\rm n}} - V_{\rm n}) = 0,$$

where $v_{i,{\rm n}}$ is the normal projection of velocity of the ith phase and $V_{\rm n}$ is the normal velocity of the free surface. Then, the absence of the mass flux through the surface $z = \alpha_1\xi + \alpha_2\xi = \xi(t, x, y)$ common for both phases acquires the form

$$\alpha_1(v_{1,{\rm n}} - V_{\rm n}) + \alpha_2(v_{2,{\rm n}} - V_{\rm n}) = 0$$

or

$$\alpha_1 v_{1,{\rm n}} + \alpha_2 v_{2,{\rm n}} = V_{\rm n} \qquad \text{for} \quad z = \xi(t, x, y). \tag{1.4}$$

Here $\alpha_1 v_{1,{\rm n}} + \alpha_2 v_{2,{\rm n}}$ is the normal projection of the volume velocity of the mixture. Condition (1.4) is the kinematic condition for the case considered. The equality of momentum fluxes on the part of the free surface $z = \alpha_i\xi$ occupied by the ith phase is $\alpha_i(\rho_i^0 v_{i,{\rm n}}(v_{i,{\rm n}} - V_{\rm n}) + P_i - P_{\rm atm}) = 0$ [3]. With allowance for the absence of the flow of the ith phase, this equality is equivalent to $\alpha_i(P_i - P_{\rm atm}) = 0$. Ignoring the normal components of viscous stresses on interfaces, we obtain the condition $\alpha_1 P_1 + \alpha_2 P_2 = P_{\rm atm}$ for the common surface of the mixture $z = \alpha_1\xi + \alpha_2\xi = \xi(t, x, y)$. Substituting relations (1.2) into this equation, we obtain the dynamic condition for the two-phase mixture:

$$p' - (\alpha_1\rho_1^0 + \alpha_2\rho_2^0)gz = 0 \qquad \text{for } z = \xi(t, x, y). \tag{1.5}$$

The quantity $P = \alpha_1 P_1 + \alpha_2 P_2 = P_{\rm atm} + p' - (\alpha_1\rho_1^0 + \alpha_2\rho_2^0)gz$ is the pressure in the mixture. Therefore, the dynamic condition can be formulated as the equality of the pressure in the mixture and the atmospheric pressure on the free surface: $P = P_{\rm atm}$. Similar conditions on the interface "liquid–liquid with suspended particles" can be found in [2]. If the constant term ρ_i^0 is retained in the kinematic conditions, then the condition for the mixture (1.4) is formulated for the mean-mass velocity of the mixture $(\rho_1\boldsymbol{v}_1 + \rho_2\boldsymbol{v}_2)/(\rho_1 + \rho_2)$. In this case, it follows from the solution of the wave problem [5] that the mixture performs nondecaying wave motions with the gravity-wave frequency as an ideal liquid.

Assuming that there is no mass flux of the mixture through the horizontal surface of the solid base, we use the no-slip condition for each phase at the bottom [3]:

$$v_{i,{\rm n}} = 0 \qquad (i = 1,\ 2) \qquad \text{for} \quad z = -l. \tag{1.6}$$

The equations in the layer of the mixture (1.3) and the boundary conditions (1.4)–(1.6) form a closed system for the unknowns $\boldsymbol{v}_i$, α_i, p', and ξ.

2. Boundary-Value Problem for Plane Waves. We consider a plane–parallel wave motion of the liquid in the plane xz. All the quantities depend only on the variables t, x, and z and $\boldsymbol{v}_i = (v_{i,x}, 0, v_{i,z})$. Let a wave with a wavelength λ ($k = 2\pi/\lambda$ is the wavenumber) propagate over the free surface in the positive direction of the x axis. The wavelength is much greater than the characteristic size of the disperse-phase particles: $\lambda \gg a$. For system (1.3) to describe the wave motion of the mixture, we introduce a wave perturbation of concentration of the disperse phase

$$\alpha_1 = 1 - \alpha_0 - \alpha'(t, x, z), \qquad \alpha_2 = \alpha_0 + \alpha'(t, x, z). \tag{2.1}$$

Here α_0 and α' are the concentration of the disperse phase in the quiescent layer of the mixture and its wave perturbation, respectively. We assume that the disperse phase is uniformly distributed in the undisturbed layer of the mixture, i.e., $\alpha_0 = {\rm const}$. Then, to determine the unknown α_1 and α_2, one has to find $\alpha'(t, x, z)$.

We introduce the following dimensionless variables and quantities:

$$
\begin{gathered}
t^* = kct, \quad x^* = kx, \quad z^* = kz, \quad \zeta = k\xi, \quad h = kl, \\
\boldsymbol{u}_i = \boldsymbol{v}_i/c, \quad p = p'/(\rho^0 c^2), \quad r = R/(\rho^0 ck), \quad \mu_i = \rho_i^0/\rho^0, \quad \gamma = \alpha'/\alpha_0.
\end{gathered} \tag{2.2}
$$

Here $\rho^0 = (1-\alpha_0)\rho_1^0 + \alpha_0\rho_2^0$ is the density of the quiescent mixture and c is the phase velocity of the wave to be determined ($ck = \omega$ is the wave frequency); the asterisk denotes dimensionless quantities (in what follows, the asterisk is omitted). Substituting relations (2.1) and dimensionless quantities (2.2) into Eqs. (1.3) and boundary conditions (1.4)–(1.6), we obtain the following boundary-value problem.

The following equations are valid in the region occupied by the mixture:

$$
\begin{gathered}
-\frac{\partial\gamma}{\partial t} + \Big(\frac{1}{\alpha_0} - 1 - \gamma\Big)\Big(\frac{\partial u_{1,x}}{\partial x} + \frac{\partial u_{1,z}}{\partial z}\Big) - u_{1,x}\,\frac{\partial\gamma}{\partial x} - u_{1,z}\,\frac{\partial\gamma}{\partial z} = 0, \\
\frac{\partial\gamma}{\partial t} + (1+\gamma)\Big(\frac{\partial u_{2,x}}{\partial x} + \frac{\partial u_{2,z}}{\partial z}\Big) + u_{2,x}\,\frac{\partial\gamma}{\partial x} + u_{2,z}\,\frac{\partial\gamma}{\partial z} = 0, \\
\mu_1\Big(\frac{\partial u_{1,x}}{\partial t} + u_{1,x}\,\frac{\partial u_{1,x}}{\partial x} + u_{1,z}\,\frac{\partial u_{1,x}}{\partial z}\Big) + \frac{\partial p}{\partial x} - r\alpha_0(1+\gamma)(u_{2,x} - u_{1,x}) = 0, \\
\mu_1\Big(\frac{\partial u_{1,z}}{\partial t} + u_{1,x}\,\frac{\partial u_{1,z}}{\partial x} + u_{1,z}\,\frac{\partial u_{1,z}}{\partial z}\Big) + \frac{\partial p}{\partial z} - r\alpha_0(1+\gamma)(u_{2,z} - u_{1,z}) = 0, \\
\mu_2\Big(\frac{\partial u_{2,x}}{\partial t} + u_{2,x}\,\frac{\partial u_{2,x}}{\partial x} + u_{2,z}\,\frac{\partial u_{2,x}}{\partial z}\Big) + \frac{\partial p}{\partial x} + r\alpha_0\Big(\frac{1}{\alpha_0} - 1 - \gamma\Big)(u_{2,x} - u_{1,x}) = 0, \\
\mu_2\Big(\frac{\partial u_{2,z}}{\partial t} + u_{2,x}\,\frac{\partial u_{2,z}}{\partial x} + u_{2,z}\,\frac{\partial u_{2,z}}{\partial z}\Big) + \frac{\partial p}{\partial z} + r\alpha_0\Big(\frac{1}{\alpha_0} - 1 - \gamma\Big)(u_{2,z} - u_{1,z}) = 0.
\end{gathered} \tag{2.3}
$$

The following boundary conditions are set on the free surface $z = \zeta(t,x)$:

$$
\frac{\partial\zeta}{\partial t} = \alpha_0\Big(\frac{1}{\alpha_0} - 1 - \gamma\Big)u_{1,z} + \alpha_0(1+\gamma)u_{2,z} - \alpha_0\Big[\Big(\frac{1}{\alpha_0} - 1 - \gamma\Big)u_{1,x} + (1+\gamma)u_{2,x}\Big]\frac{\partial\zeta}{\partial x}; \tag{2.4}
$$

$$
p - \nu^2\zeta[1 - \alpha_0\gamma(\mu_1 - \mu_2)] = 0, \qquad \nu^2 = g/(kc^2). \tag{2.5}
$$

The conditions on the bottom $z = -h$ are

$$
u_{i,z} = 0 \qquad (i = 1,\ 2). \tag{2.6}
$$

Equations (2.3) and the boundary conditions (2.4)–(2.6) constitute a nonlinear boundary-value problem for determining the velocities of the wave motion of the phases, the perturbations of pressure and concentration, and the shape of the free surface.

We consider the linear variant of problem (2.3)–(2.6). We assume that the amplitude of the surface wave is small as compared to its length. Then, the boundary conditions on the free surface $z = \zeta(t,x)$ can be reduced to the conditions on the fixed surface $z = 0$. To this end, we expand all the functions that enter Eqs. (2.4) and (2.5) into the Maclaurin series in the vicinity of $z = 0$, for example,

$$
u_{1,z}(t,x,\zeta) = u_{1,z}(t,x,0) + \frac{\partial u_{1,z}}{\partial z}\Big|_{z=0}\zeta + \frac{1}{2}\,\frac{\partial^2 u_{1,z}}{\partial z^2}\Big|_{z=0}\zeta^2 + \ldots\ .
$$

In addition, it follows from nondimensionalization of (2.2) that the velocities of the wave motion of the phases and wave perturbations are of the same order as ζ, i.e., they are small.

Taking into account the smallness of unknowns that enter system (2.3)–(2.5), we retain only linear terms with respect to these unknowns in Eqs. (2.3) and boundary conditions (2.4) and (2.5) expanded into a series in the vicinity of $z = 0$. Thus, we obtain the following linear problem:

$$
\begin{gathered}
-\alpha_0\,\frac{\partial\gamma}{\partial t} + (1-\alpha_0)\Big(\frac{\partial u_{1,x}}{\partial x} + \frac{\partial u_{1,z}}{\partial z}\Big) = 0, \qquad \frac{\partial\gamma}{\partial t} + \frac{\partial u_{2,x}}{\partial x} + \frac{\partial u_{2,z}}{\partial z} = 0, \\
\mu_1\,\frac{\partial u_{1,x}}{\partial t} + \frac{\partial p}{\partial x} - r\alpha_0(u_{2,x} - u_{1,x}) = 0, \qquad \mu_1\,\frac{\partial u_{1,z}}{\partial t} + \frac{\partial p}{\partial z} - r\alpha_0(u_{2,z} - u_{1,z}) = 0, \\
\mu_2\,\frac{\partial u_{2,x}}{\partial t} + \frac{\partial p}{\partial x} + r(1-\alpha_0)(u_{2,x} - u_{1,x}) = 0, \qquad \mu_2\,\frac{\partial u_{2,z}}{\partial t} + \frac{\partial p}{\partial z} + r(1-\alpha_0)(u_{2,z} - u_{1,z}) = 0.
\end{gathered} \tag{2.7}
$$

For $z = 0$, we have

$$\frac{\partial \zeta}{\partial t} = (1 - \alpha_0)u_{1,z} + \alpha_0 u_{2,z}; \tag{2.8}$$

$$p - \nu^2 \zeta = 0. \tag{2.9}$$

For $z = -h$, the conditions remain unchanged, i.e., in the form (2.6).

The solution of problem (2.7)–(2.9) has to satisfy some requirements. The relative motion of the phases and the forces of interphase interaction are responsible for the dissipative process — decay of the wave motion. In the absence of the disperse phase ($\alpha_0 = 0$) or in the case of identical true densities of the phases ($\rho_1^0 = \rho_2^0$), the solution of the problem should reduce to the known wave solutions for liquids [4]. The components of phase velocities $u_{1,z}$ and $u_{2,z}$ should satisfy the boundary conditions at the bottom (2.6). In the case of propagation of progressive waves over the free surface of the layer, the solution of system (2.7) that satisfies the above requirements should be sought in the following form:

$$\begin{aligned} u_{i,x} &= (\cosh\,(z+h)/\sinh\,h)(A_i \sin(x-t) + B_i \cos(x-t)) \exp(-bt), \\ u_{i,z} &= (\sinh\,(z+h)/\sinh\,h)(C_i \sin(x-t) + D_i \cos(x-t)) \exp(-bt), \\ p &= (\cosh\,(z+h)/\sinh\,h)(K \sin(x-t) + L \cos(x-t)) \exp(-bt), \\ \gamma &= (\cosh\,(z+h)/\sinh\,h)(M \sin(x-t) + N \cos(x-t)) \exp(-bt). \end{aligned} \tag{2.10}$$

Here $b = \beta/\omega$ is the dimensionless damping factor of the wave, β is the dimensional damping factor, and the coefficients A_i, B_i, C_i, D_i, K, L, M, and N are constants to be found. Substituting expressions (2.10) into (2.7), we obtain a system of twelve linear homogeneous equations for determining the unknown coefficients:

$$-(1-\alpha_0)B_1 + (1-\alpha_0)C_1 + b\alpha_0 M - \alpha_0 N = 0, \quad (1-\alpha_0)A_1 + (1-\alpha_0)D_1 + \alpha_0 M + b\alpha_0 N = 0,$$

$$-B_2 + C_2 - bM + N = 0, \qquad A_2 + D_2 - M - bN = 0,$$

$$(\alpha_0 r - b\mu_1)A_1 - \alpha_0 r A_2 + \mu_1 B_1 - L = 0, \quad -\mu_1 A_1 + (\alpha_0 r - b\mu_1)B_1 - \alpha_0 r B_2 + K = 0,$$

$$(\alpha_0 r - b\mu_1)C_1 - \alpha_0 r C_2 + \mu_1 D_1 + K = 0, \quad -\mu_1 C_1 + (\alpha_0 r - b\mu_1)D_1 - \alpha_0 r D_2 + L = 0,$$

$$-(1-\alpha_0)rA_1 + ((1-\alpha_0)r - b\mu_2)A_2 + \mu_2 B_2 - L = 0, \quad -\mu_2 A_2 - (1-\alpha_0)rB_1 + ((1-\alpha_0)r - b\mu_2)B_2 + K = 0,$$

$$-(1-\alpha_0)rC_1 + ((1-\alpha_0)r - b\mu_2)C_2 + \mu_2 D_2 + K = 0, \quad -\mu_2 C_2 - (1-\alpha_0)rD_1 + ((1-\alpha_0)r - b\mu_2)D_2 + L = 0.$$

This system is a system of rank 10; hence, two unknown constants are free. The coefficients at K and L do not enter the matrix forming the rank of the system; therefore, we consider them to be free. We determine the remaining unknown constants as the solution of the system

$$\begin{gathered} D_i = -A_i, \quad B_i = C_i, \quad M = 0, \quad N = 0, \quad A_i = m_i K + n_i L, \quad B_i = -n_i K + m_i L, \\ m_1 = \frac{1}{1+b^2}\Big(1 + \frac{\mu_1\mu_2^2(1-\mu_1)(1+b^2)}{d}\Big), \quad m_2 = \frac{1}{1+b^2}\Big(1 + \frac{\mu_1^2\mu_2(1-\mu_2)(1+b^2)}{d}\Big), \\ n_1 = \frac{1}{1+b^2}\Big(-b + \frac{\mu_2(1-\mu_1)(r - b\mu_1\mu_2)(1+b^2)}{d}\Big), \\ n_2 = \frac{1}{1+b^2}\Big(-b + \frac{\mu_1(1-\mu_2)(r - b\mu_1\mu_2)(1+b^2)}{d}\Big), \qquad d = \mu_1^2\mu_2^2 + (r - b\mu_1\mu_2)^2. \end{gathered} \tag{2.11}$$

It follows from relations (2.10) and (2.11) that the perturbation of concentration of the disperse phase equals zero in the linear approximation, and it is a quantity of a higher order of smallness than velocity and pressure perturbations. Thus, Lobov et al. [2] assumed that the concentration was constant in their formulation of the problem.

To determine the shape of the free surface, we have to substitute the relations found for $u_{i,z}$ into Eq. (2.8) and integrate the latter. As a result, we obtain

$$\begin{gathered} \zeta = [(s_1 K + s_2 L)\sin(x-t) + (-s_2 K + s_1 L)\cos(x-t)]\exp(-bt)/(1+b^2), \\ s_1 = \frac{1-b^2}{1+b^2} + \frac{(br + (1+b^2)\mu_1\mu_2)(\sigma - \mu_1\mu_2)}{d}, \quad s_2 = -\frac{2b}{1+b^2} + \frac{(r - 2b\mu_1\mu_2)(\sigma - \mu_1\mu_2)}{d}. \end{gathered} \tag{2.12}$$

Hereinafter, $\sigma = (1 - \alpha_0)\mu_2 + \alpha_0\mu_1$. As for the usual surface waves [4], the free coefficients K and L can be determined from additional initial data.

Using the solution (2.10)–(2.12), we can easily find the wave perturbations and the shape of the free surface for a dissipationless motion of the mixture, i.e., for $r = 0$. In this case, we have $b = 0$, and relations (2.10) and (2.12) coincide with the classical ones [4]. Note also that the solution of system (2.7) cannot be found in the form of monochromatic waves [only with terms $\sin(x-t)$ or $\cos(x-t)$], as it was suggested in [2], because in this case the system of algebraic equations for the coefficients has the trivial solution only.

3. Phase Velocity, Frequency, and Damping Factor of the Wave. We find the dispersion relation and the equation for the damping factor of the wave. Substituting the expressions for pressure and free surface perturbations into the dynamic condition (2.9) and equating the coefficients at $\sin(x-t)$ and $\cos(x-t)$, we obtain

$$K(1+b^2)\coth h = \nu^2(s_1K + s_2L), \qquad L(1+b^2)\coth h = \nu^2(-s_2K + s_1L).$$

The condition of existence of a nontrivial solution of this system for K and L has the form $(1 + b^2 - s_1\nu^2\tanh h)^2 + (s_2\nu^2\tanh h)^2 = 0$, which is equivalent to the system

$$s_1 = (1+b^2)/(\nu^2\tanh h), \qquad s_2 = 0, \tag{3.1}$$

which allows one to determine the unknown phase velocity and damping factor of the wave. In addition, using the resultant coefficients (3.1), we can write a refined expression for the shape of the free surface:

$$\zeta = \coth h[K\sin(x-t) + L\cos(x-t)]\exp(-bt)/\nu^2.$$

To find the phase velocity and damping factor, we have to write the quantities that enter Eqs. (3.1) in a dimensional form. The equations become too cumbersome. To avoid this, we introduce auxiliary quantities, which have the dimension of velocity: $r_1 = cr = R/(\rho^0 k)$, $b_1 = cb = \beta/k$, and $\nu_1^2 = c\nu^2 = g/k$. With the use of these quantities, after simple transformations, system (3.1) becomes

$$(c+b_1)^2(r_1 - 2\mu_1\mu_2 b_1) = \nu_1^2\tanh h, \quad 2b_1(\mu_1^2\mu_2^2(c^2+b_1^2) + r_1(r_1 - 2\mu_1\mu_2 b_1)) = r_1(\sigma - \mu_1\mu_2)\tanh h.$$

Solving the resultant system with respect to c and b_1, we obtain the following equations for determining the phase velocity and the damping factor of the wave:

$$c^2 = \sigma\nu_1^2\tanh h/(\mu_1\mu_2) + b_1(3b_1 - 2r_1/(\mu_1\mu_2)); \tag{3.2}$$

$$8\mu_1^2\mu_2^2 b_1^3 - 8\mu_1\mu_2 r_1 b_1^2 + 2(r_1^2 + \sigma\mu_1\mu_2\nu_1^2\tanh h)b_1 - (\sigma - \mu_1\mu_2)r_1\nu_1^2\tanh h = 0. \tag{3.3}$$

Equation (3.2) can be written in the form

$$c^2 = c_g^2 + c_d^2 + c_r^2, \tag{3.2a}$$

where

$$c_g^2 = \nu_1^2\tanh h = \frac{g}{k}\tanh(kl), \quad c_d^2 = \frac{\sigma - \mu_1\mu_2}{\mu_1\mu_2}\nu_1^2\tanh h = \frac{\alpha_0(1-\alpha_0)(\rho_1^0 - \rho_2^0)^2 g\tanh(kl)}{\rho_1^0\rho_2^0 k},$$

$$c_r^2 = b_1\Big(3b_1 - \frac{2r_1}{\mu_1\mu_2}\Big) = \frac{\beta}{k^2}\Big(3\beta - \frac{2R\rho^0}{\rho_1^0\rho_2^0}\Big).$$

Equation (3.2a) easily yields an expression for the wave frequency $\omega^2 = c^2k^2$. The value of c_g^2 is equal to the squared phase velocity of the gravity wave [4], $c_d^2 \geqslant 0$ is the increment to the phase velocity due to the presence of the disperse phase, and c_r^2 is the increment caused by the forces of interphase interaction. The function $c_r^2(b_1)$ acquires negative values for $0 < b_1 < 2r_1/(3\mu_1\mu_2)$ and positive values for $b_1 > 2r_1/(3\mu_1\mu_2)$. The value of the damping factor $b_1 = 2r_1/(3\mu_1\mu_2)$ is critical, since the forces of interphase interaction do not affect wave propagation in this case. The value of c_r^2 is minimum at $b_1 = r_1/(3\mu_1\mu_2)$: $\min c_r^2 = -r_1^2/(3\mu_1^2\mu_2^2)$. In what follows, we introduce the notation $|\min c_r^2| = r_1^2/(3\mu_1^2\mu_2^2) = c_{\min}^2$.

The condition of existence of steady waves is the nonnegative value of the phase velocity squared: $c^2 \geqslant 0$. This condition is equivalent to the nonnegative value of the squared polynomial for b_1 in the right side of Eq. (3.2) and, hence, to the nonnegative value of the discriminant of this polynomial, which is satisfied for $\sigma\nu_1^2\tanh h \geqslant r_1^2/(3\mu_1\mu_2)$. This condition can be written as a restriction on the wavelength:

$$3gk\tanh(kl)(\rho_1^0\rho_2^0 + \alpha_0(1-\alpha_0)(\rho_1^0 - \rho_2^0)^2)/R^2 \geqslant 1. \tag{3.4}$$

Condition (3.4) can be written using new dimensionless variables:

$$W = (c_g^2 + c_d^2)/c_{\min}^2 \geqslant 1. \tag{3.4a}$$

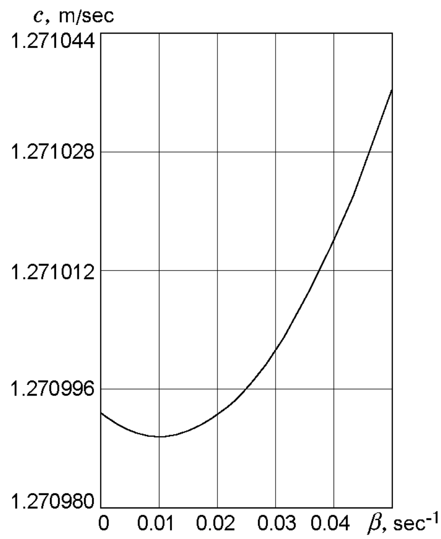

Fig. 1

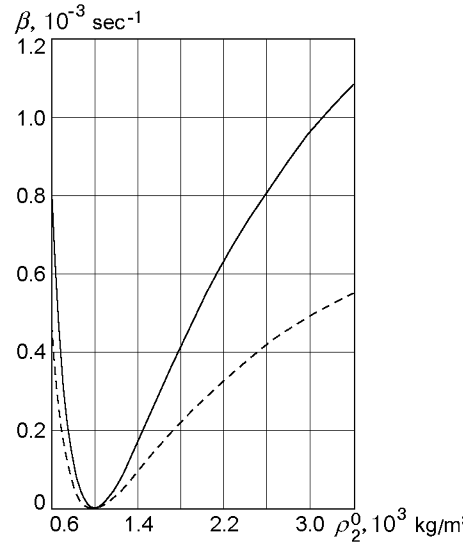

Fig. 2

Fig. 1. Phase velocity of the wave c as a function of the damping factor β for $\rho_2^0 = 1500$ kg/m^3 and $\alpha_0 = 0.1$.

Fig. 2. Damping factor β as a function of the true density of the second phase ρ_2^0 for $\alpha_0 = 0.1$ (solid curve) and 0.05 (dashed curve).

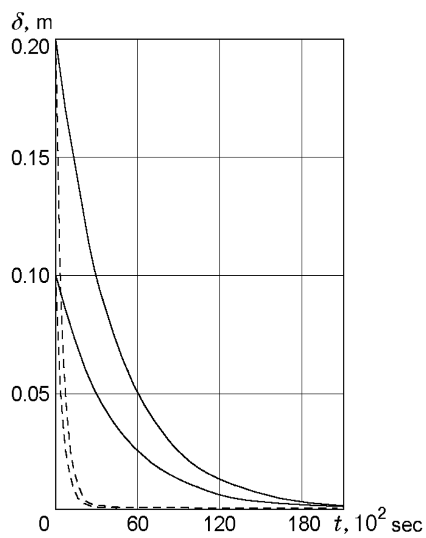

Fig. 3. Wave amplitude δ versus the time t for $\alpha_0 = 0.1$ and $\rho_2^0 = 1500$ (solid curves) and 500 kg/m^3 (dashed curves).

For $W = 1$, the phase velocity of the wave acquires the minimum value

$$\min c^2 = c_g^2 + c_d^2 - c_{\min}^2 = \sigma\nu_1^2 \tanh h/(\mu_1\mu_2) - r_1^2/(3\mu_1^2\mu_2^2).$$

Before finding the solution of the cubic equation (3.3), we note that all the coefficients of this equation satisfy the Hurwitz criterion of stability [6]. We solve Eq. (3.3) using the Cardano formula [7]

$$b_1 = [-\chi/2 + (\chi^2/4 + \psi^3/27)^{1/2}]^{1/3} + [-\chi/2 - (\chi^2/4 + \psi^3/27)^{1/2}]^{1/3} + r_1/(3\mu_1\mu_2), \qquad (3.5)$$

where

$$\psi = (3\sigma\mu_1\mu_2\nu_1^2 \tanh h - r_1^2)/(12\mu_1^2\mu_2^2) = c_{\min}^2(W-1)/4 = (1/4)\min c^2,$$

$$\chi = r_1[2r_1^2 - 9\mu_1\mu_2(\sigma - 3\mu_1\mu_2)\nu_1^2 \tanh h]/(216\mu_1^3\mu_2^3) = (3c_{\min}^2)^{3/2}[1 + 3(3c_g^2/c_{\min}^2 - W)/2]/108.$$

It is known that the number of roots depends on the sign of the quantity $Q = \chi^2/4 + \psi^3/27$ [7]. In the case considered, by virtue of (3.4a), we have $Q > 0$ for $W > 1$, and hence, formula (3.5) yields one real and two complex-conjugate roots. Therefore, the solution of the equation for $W > 1$ is only one root, which is the real root (3.5). The quantity Q can be equal to zero only if $\psi = 0$ and $\chi = 0$ simultaneously, which occurs at $W = 1$ and $c_g^2 = c_{\min}^2/9$ (or $c_d^2 = 8c_{\min}^2/9$). The equality $Q = 0$ corresponds to the minimum phase velocity and the damping factor $b_1 = r_1/(3\mu_1\mu_2)$; $c^2 = \min c^2 = 0$.

We give the final dimensional expressions for the pressure in the mixture and the shape of the free surface:

$$P = P_{\text{atm}} - \rho^0 gz + \rho^0 c^2 \cosh\,(k(z+l))[K \sin(k(x-ct)) + L\cos(k(x-ct))]\exp(-\beta t)/\sinh\,(kl),$$

$$\xi = c^2 \coth\,(kl)[K\sin(k(x-ct)) + L\cos(k(x-ct))]\exp(-\beta t)/g,$$

Here c^2 is determined by formula (3.2).

4. Example of Calculations. To illustrate the obtained solutions (3.2a) and (3.5), we calculated the wave-motion parameters in a layer of the mixture of thickness $l = 100$ m, which was caused by propagation of a wave 1 m long over the free surface. It was assumed that the forces of interphase interaction were represented only by the Stokes force of viscous friction, i.e., $R = 9\eta/(2a^2)$. For undeformable spheres of radius $a = 10^{-2}$ m, for $\eta = 1.004 \cdot 10^{-3}$ kg/(m $\cdot$ sec), the empirical coefficient was $R = 45$ kg/(m$^3 \cdot$ sec).

Figure 1 shows the phase velocity as a function of the damping factor of the wave. As is shown above, with increasing β, the phase velocity first decreases to $c_{\min}$ (in our case, $c_{\min} \approx 1.27099$ m/sec), which corresponds to $\beta = R\rho^0/(3\rho_1^0\rho_2^0)$ (in our case, $\beta \approx 0.01$ sec^{-1}) and then increases; the action of interphase friction is not manifested for $\beta = 2R\rho^0/(3\rho_1^0\rho_2^0)$ (in our case, $\beta \approx 0.02$ sec^{-1}).

Figure 2 shows the damping factor as a function of density of admixtures. It follows from Fig. 2 that the wave decays much more rapidly if the disperse phase is less dense than the carrier medium.

From the resultant expression for the free surface shape, we can derive a formula for the wave amplitude. For $K = L$, the amplitude is determined as

$$\delta = \max \xi(t,x) - \min \xi(t,x) = \xi(t,x_0) - \xi(t,x_1) = 2\sqrt{2}\coth\,(kl)K\exp(-\beta t)/\nu^2,$$

$$x_0 = \lambda/8 + ct, \qquad x_1 = 5\lambda/8 + ct.$$

Using this expression and Eq. (3.5), we can readily verify that the time of wave decay for admixtures of density $\rho_2^0 < \rho_1^0$ is several times smaller than in the case $\rho_2^0 > \rho_1^0$.

Figure 3 shows the decrease in the wave amplitude for $K = 1$ and $\alpha_0 = 0.1$. The initial values of the amplitude are 0.1 and 0.2 m. It follows from Fig. 3 that the waves on the free surface of the mixture with particles less heavy that the carrier liquid decay much more rapidly than waves with heavier particles. Thus, for the case illustrated in Fig. 3, the decay time differs approximately by a factor of 7.

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